(8 pages) **Reg. No. :** ....

Code No. : 8154 **Sub. Code : VMAC 11**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2024.

First Semester

Mathematics – Core

GROUP THEORY

(For those who joined in July 2024 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. If G is a group and H is a subgroup of index 2 in G then

   (a) H is a normal subgroup of G

   (b) H is a belian in G

   (c) $aHa^{-1} \neq H, \forall a \in G$

   (d) None of these

2. Let N be a subgroup of G and every left coset of N in G is the right coset of N in G which one of the following is false?

   (a) N is a normal subgroup of G

   (b) $NaNb = Nab$, $\forall a, b \in G$

   (c) G/N is a group

   (d) None of these

3. Let G be a group and $O(G) = 36$. Let H be a subgroup and $O(H) = 9$. Then H contains a normal subgroup of order.

   (a) 3 or 4 (b) 5 or 7

   (c) 3 or 6 (d) 3 or 9

4. A group G is solvable if there exist subgroups $G = N_0 \supset N_1 \supset ...... \supset N_r = \{e\}$ such that

   (a) Each N_i is normal in N_{i-1}

   (b) N_{i-1}/N_i is abelian

   (c) Both (a) and (b) are true

   (d) None of these

5. Let G be the group and ϕ is an automorphism of G. If $a \in G$ is of order $0(a) > 0$, then $0(\phi(a)) =$ ————.

   (a) 0 (b) 1

   (c) a (d) a

6. If G is a group of order 99 and H is a subgroup of G of order 11 then $i(H) =$ ________.

(a) 9 (b) 9!

(c) 11 (d) 11!

7. Product of two odd permutation is ________ permutation.

(a) Odd

(b) Even

(c) Both (a) and (b)

(d) None of these

8. Let $a \in z(G)$ then

(a) $N(a) = G$ (b) $N(a) \neq G$

(c) $O(z) = P^n$ (d) None of these

9. If $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ the $\alpha\beta =$ ________.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

10. S_{p^k} has a P -sylow subgroup of order

(a) $P^{n(k)}$ (b) P^k

(c) P (d) None of these

11. Let G be a group and $0(G) = 72$. Then

(a) G must have a non-trivial normal subgroup

(b) G is simple

(c) G is not having any normal subgroup

(d) None of these

12. The number of P-sylow subgroups in G, for a given prime is of the form ________.

(a) $1+KP$ (b) $1-KP$

(c) KP (d) $\frac{1+K}{P}$

13. Let G be a finite abelian group. Then G is isomophic to the direct product of its.

(a) Subgroups (b) Normal subgroups

(c) Subsets (d) Sylow subgroups

14. The Number of non-isomorphic abelian groups of order P^n is

(a) $P(n)$ (b) P^3

(c) P (d) None of these

[P.T.O.]

15. The number of non-isomorphic abelian groups of order 2^4 is ————.

(a) 4 (b) 5

(c) 7 (d) 1

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).
Each answer should not exceed 250 words.

16. (a) HK is a subgroup of G iff $HK = KH$.

Or

(b) If ϕ is a homomorphism of G into $\overline{G}$ with kernel K, then K is a normal subsgroup of $\overline{G}$.

17. (a) Prove that $I(G) \approx G/z$, where $I(G)$ is the group of inner automorphisms of $\overline{G}$ and Z is the center of $\overline{G}$.

Or

(b) Prove that a subgroup of a solvable group is solvable.

18. (a) Prove that A_n is a normal subgroup of index z in S_n.

Or

(b) If $O(G) = P^2$, where P is a prime number show that $\overline{G}$ is abelian.

19. (a) Prove that $n(k) = 1 + P + ...P^{k-1}$.

Or

(b) If $P^m / O(G)$, $P^{m+1} O(G)$, then prove that $\overline{G}$ has a subgroup of order P^m.

20. (a) Suppose that $\overline{G}$ is the internal direct product of $N_1, N_2, ...N_n$. Then for $c \neq d$, prove that $N_i \cap N_j = (e)$ and if $a \in N_i, b \in N_j$, then $ab = ba$.

Or

(b) If G and G' are isomorphic abelian groups then for every integer S, show that $G(s)$ and $G'(s)$ are isomorphic.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)
Each answer should not exceed 600 words.

21. (a) Let ϕ be a homomorphism of $\overline{G}$ onto $\overline{G}$ with Kernel K. Prove that $G/K \approx \overline{G}$.

Or

(b) State and prove Cauchy's theorem for abelian groups.

22. (a) (i) Prove that (i) If $\overline{G}$ is a group then $A(G)$, the set of automorphisms of $\overline{G}$, is also a group.

(ii) Prove that $J(G) \equiv G/z$, where $J(G)$ is the group of inner automorphisms of G and z is the center of G.

Or

(b) (i) Let $G = S_n$, where $n \geq 5$, then $G^{(k)}$ for $k = 1, 2, \ldots\ldots$ contains every 3 cycle of S_n.

(ii) S_n is not solvable for $n \geq 5$.

23. (a) If P is a prime number and $P/O(G)$ then G has an element of $O(G)$.

Or

(b) Prove that the number of conjugate classes in S_n is P_n, the number of partitions of n.

24. (a) State and prove the second part of Sylow's theorem.

Or

(b) Prove that S_p^k has a P-sylow subgroup.

25. (a) Define the integral and external direct product of normal subgroups and show that they are isomorphic.

Or

(b) Show that every finite abelian group is the direct product of cyclic groups.

Reg. No. : ....

Code No. : 7752 **Sub. Code : WMAM 12/ VMAC 12**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2024.

First Semester

Mathematics — Core

REAL ANALYSIS – I

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. Absolute convergence of $\sum a_n$ implies ————

   (a) Divergence

   (b) Convergence

   (c) $|a_n|$ diverges

   (d) All

2. The total variation $V_f(a,b)=0$ if and only if f is ———— on $[a,b]$

(a) Continuous

(b) Constant

(c) Variable

(d) ∞

3. A series $\sum a_n$ is conditionaly convergent if $\sum a_n$ converges but $\sum |a_n|$ ————

(a) Converges

(b) Conditionaly converges

(c) Diverges

(d) Both (a) and (b)

4. If f and g belongs to $R(\alpha)$ where $\alpha \nearrow$ on $[a,b]$ then the product fg ————

(a) Does not belong to $R(\alpha)$

(b) Belongs to $R(\alpha)$

(c) Both (a) and (b)

(d) None

5. The length of the largest subinterval of the partition p is called as ————

(a) modulus of p

(b) norm of p

(c) absolute value of p

(d) All

6. If $a<b$, then $\int_a^b f\,dx =$ ———— when ever $\int_a^b f\,dx$ exists.

(a) $-\int_a^b f\,dx$

(b) $\int_a^b f\,dx$

(c) 1

(d) $-\int_b^a f\,dx$

7. If α be continuous and $f \nearrow$ on $[a,b]$, then there exists a point x_0 in $[a,b]$ such that $\int_a^b f(x)\,d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) +$ ————

(a) $\int_{x_0}^b d\alpha(x)$

(b) $f(b)\int_{x_0}^b d\alpha(x)$

(c) $f(a)\int_{x_0}^b d\alpha(x)$

(d) $\int_b^{x_0} d\alpha(x)$

8. One of the sufficient condition for the existence of the Riemann integral $\int_a^b f(x)\,dx$ is ———— on $[a,b]$.

(a) f is continuous (b) f is not continuous

(c) f is compact (d) All

9. If $f \in R$ and α a continuous function on $[a,b]$ whose derivative α' is Riemann integral on $[a,b]$ then $\int_a^b f(x)\,d\alpha(x)$ ———— $\int_a^b f(x)\,\alpha'(x)\,dx$.

(a) $=$ (b) $\neq$

(c) $<$ (d) $>$

10. A function f whose domain is $z^+ \times z^+$ is called a ———— sequence.

(a) Convergent (b) Cauchy

(c) Double (d) Single

11. The double series is said to ———— to the sum a if $\lim_{p,q \to \infty} (S(pq)) = a$.

(a) Converge (b) Diverge

(c) Oscillate (d) All

12. An infinite series of the form $a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n$ is called ————

(a) Exponential series

(b) Power series in $(z - z_0)$

(c) Trigonometric series in $(z - z_0)$

(d) Logarithmic series

13. If C is the accumulation point of S, then $\lim_{x \to c} \lim_{n \to \infty} f_n(x) =$ ————

(a) 0

(b) 1

(c) $\lim_{n \to \infty} \lim_{x \to c} f_n(x)$

(d) $\lim_{n \to \infty} f_n(c)$

14. A sequence $\{f_n\}$ is said to be ———— on S if there exists a constant $M > 0$ such that $|f_n(x) \leq M|$ for all x in S and all n

(a) Uniformly bounded

(b) Converge uniformly

(c) Both (a) and (b)

(d) Diverge

15. Let $\sum f_n(x) = f(x)$ if each f_n is continuous at a point x_0 of S, then f is ———— at x_0.

(a) Not continuous

(b) Continuous

(c) Diverge

(d) Oscillate

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let f be of bounded variation on $[a, b]$ Let $V(x) = V_f(a, x)$ if $a < x \le b, v(a) = 0$ on $[a, b]$. Then prove the following.

(i) V is an increasing function on $[a, b]$

(ii) V– f is an increasing function on $[a, b]$

Or

(b) If f is continuous on $[a, b]$ and if f' exists and is bounded in the interior, say $|f'(x)| \le A$ for all x in (a, b) then prove that f is of bounded variation on $[a, b]$.

17. (a) Assume that $c \in (a, b)$ if two of the three integrals given below exist prove that the third also exists and we have $\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha$.

Or

(b) Let $\alpha \nearrow$ on $[a, b]$ if $f \in R(\alpha)$ on $[a, b]$ then show that $|f| \in R(\alpha)$ on $[a, b]$ and $\left|\int_a^b f(x)\, d\alpha(x)\right| \le \int_a^b |f(x)|\, d\alpha(x)$.

18. (a) State and prove second fundamental theorem of integral calculus.

Or

(b) Assume that α is continuous and that $f \nearrow$ on $[a, b]$ then prove that there exists a point x_0 in $[a, b]$ such that $\int_a^b f(x)\, d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$.

19. (a) Let $a_n > 0$ then show that the product $\pi(1+a_n)$ converges iff the series $\sum a_n$ converges.

Or

(b) Show that if a series is convergent with sum S, then it is also $(C,1)$ summable with cesaro sum S.

20. (a) Assume that $f_n \to f$ uniformly on S. If each f_n is continuous at a point c of s, then prove that the limit function f is also continuous at C.

Or

(b) State and prove Dirichlet's test for uniform convergence.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

21. (a) State and prove Riemann's theorem on conditionally convergent series.

Or

(b) Let F be of bounded variation on $[a, b]$ and assume that $C \in (a,b)$ then prove that f is of bounded variation on $[a, c]$ and on $[c, b]$ and $V_f(a, b) = V_f(a, c) + V_f(c,b)$.

22. (a) Assume that $\alpha \nearrow$ on $[a,b]$. The prove that the following statement are equivalent.

(i) $f \in R(\alpha)$ on $[a,b]$

(ii) f satisfies riemann's condition with respect n ot α on $[a,b]$

(iii) $\underline{I}(f,\alpha) = \overline{I}(f,\alpha)$.

Or

(b) If $f \in R(\alpha)$ on $[a,b]$ then prove that $\alpha \in R(f)$ on $[a,b]$ and

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df(x) = f(b)\alpha(b) - f(a)\alpha(a)$$

23. (a) Assume that α is of bounded variation on $[a,b]$. Let $V(x)$ denote the total variation of α on $[a,x]$ if $a < x \le b$ and $v(a) = 0$ Let f be defined and bounded on $[a,b]$ If $f \in R(\alpha)$ on $[a,b]$ then prove that $f \in R(V)$ on $[a,b]$.

Or

(b) Discuss the theorem on change of variable in a Riemann integral.

24. (a) State and prove Merten's theorem.

Or

(b) Assume f has a continuous derivative of order $n+1$ in some open interval I containing c. Define $E_n(x)$ for x in I by $f(x) = \sum \frac{f^{(k)}(c)}{k!}(x-c)^k + E_n(x)$. Then prove that $E_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt$.

25. (a) State and prove the theorem on Cauchy condition for uniform convergence.

Or

(b) Discuss and prove three examples of sequences of real valued functions.

(8 pages) **Reg. No. :** ....

Code No. : 7753 **Sub. Code : WMAM 13/ VMAC 13**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2024.

First Semester

Mathematics – Core

ORDINARY DIFFERENTIAL EQUATIONS

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. The characteristic polynomial of $y''+y'-2y=0$ is ————

(a) r^2+r+2 (b) r^2+r-2

(c) $(r^2+r-2)y=0$ (d) r_1, r_2

2. Two solutions φ_1, φ_2 of $L(y)=0$ are linearly dependent iff $W(\varphi_1 \varphi_2)=$ ————

(a) 0 (b) 1

(c) ∞ (d) $\neq 0$

3. The value of the Wronksian is ———— if $\varphi_1=\cos x$ and $\varphi_2=\sin x$.

(a) 0 (b) –1

(c) 1 (d) 2

4. The roots of the characteristic polynomial of the equation $y'''-y'=0$ are ————

(a) 0, 1, –1 (b) 0, 1, 1

(c) 1, 1, 1 (d) 1, –1, –1

5. The real valued solution of $y''+y=0$ is ————

(a) c_1+c_2x (b) $c_1e^x+c_2e^{-x}$

(c) $c_1\cos x+c_2\sin x$ (d) $c_1x+c_2x^{-1}$

6. The particular solution of the equation $y''+4y=\cos x$ is ————

(a) $\frac{1}{3}\sin x$ (b) $\frac{1}{3}x$

(c) $\frac{1}{3}$ (d) $\frac{1}{3}\cos x$

7. The linear differential equation $L(y)=b(x)$ is said to be non-homogeneous equation if $b(x)$ ________

(a) $=0$ (b) $\neq 0$

(c) >0 (d) <0

8. The n functions $\varphi_1, \varphi_2, ..., \varphi_n$ defined on an interval I are said to be ________ if the only constants $c_1...c_n$ such that $c_1\varphi_1(x)+...+c_n\varphi_n(x)=0$ for all x in I are the constants $c_1=c_2=...=c_n=0$.

(a) linearly independent

(b) linearly dependent

(c) cannot say

(d) both (a) and (b)

9. The value of the Legendre polynomial $p_1(x)$ is ________

(a) 1 (b) 0

(c) x (d) x^2

10. The singular point and its nature of the equation $x^2y''-5y'+3x^2y=0$ is ________

(a) $x=0$, regular (b) $x=0$, not regular

(c) $x=1$, not regular (d) No singular point

11. A point x_0 such that $a_0(x_0)=0$ is called ________ point of the equation $a_0(x)y^{(n-1)}+...a_n(x)y=0$.

(a) regular (b) particular

(c) not regular (d) singular

12. The origin $x_0=0$ is ________ for the equation $x^2y''-y'-\frac{3}{4}y=0$.

(a) singular point (b) regular singular

(c) irregular singular (d) analytic

13. The solution of $y'=y^2$ with $\varphi(1)=-1$ is ________

(a) $\frac{1}{x}$ (b) x

(c) x^2 (d) $-\frac{1}{x}$

14. The equation $(x^2+xy)dx+xy\,dy=0$ is ________

(a) exact (b) not exact

(c) can't say (d) both (a) and (b)

15. The Lipschitz constant for the function $f(x,y)=4x^2+y^2 ons$, $|x|\leq 1$, $|y|\leq 1$

(a) 2 (b) 1

(c) 4 (d) 3

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If φ_1, φ_2 are two solutions of $L(y)=0$ on an interval I containing a point x_0 then prove that $W(\varphi_1,\varphi_2)=e^{-a_1(x-x_0)}\,W(\varphi_1,\varphi_2)(x_0)$.

Or

(b) Compute the solution of the initial value problem $y''-2y'-3y=0$; $y(0)=0$, $y'(0)=1$.

17. (a) Let $\varphi_1,\varphi_2,\ldots\varphi_n$ be linearly independent solution of $L(y)=0$ on an interval I if $c_1,\ldots c_n$ are any constants then prove that $\varphi=c_1\varphi_1+\ldots+\varphi_n c_n$ is a solution.

Or

(b) Find the solutions of $y'''-y'=x$.

18. (a) Find the solution φ for the equation $y''-\frac{2}{x^2}y=x\ (0<x<\infty)$. Given $\varphi_1=x^2$, $\varphi_2=x^{-1}$.

Or

(b) Verify $\varphi_1=x^3$ satisfy the equation $x^2y''-7xy'+15y=0$ $(x>0)$ and find $\varphi_2=(x)$.

19. (a) Prove that $J_0'(x)=-J_1(x)$.

Or

(b) Calculate the roots of the indicial equation of $x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0$.

20. (a) Verify whether the equation $y'=\frac{-e^x}{e^y(y+1)}$ is exact or not.

Or

(b) State and prove the theorem on Lipschitz condition.

PART C — (5 × 8 = 40 marks)

Answer ALL the questions, choosing either (a) or (b).

Each answer should not exceed 600 words.

21. (a) Let φ be any solution of $L(y)=y''+a_1y'+a_2y=0$ on an interval I containing x_0. The prove that for all x in I, $\|\varphi(x_0)\|e^{-k|x-x_0|}\le\|\varphi(x)\|\le\|\varphi(x_0)\|e^{k|x-x_0|}$ where $\|\varphi(x)\|=\left[|\varphi(x)|^2+|\varphi'(x)|^2\right]^{\frac{1}{2}}$, $k=1+|a_1|+|a_2|$.

Or

(b) Prove that two solutions φ_1, φ_2 of $L(y)=0$ are linearly independent on I if and only if $W(\varphi_1, \varphi_2)(x) \neq 0$ for all x in I.

22. (a) Compute three linearly independent solutions and the Wronskian for the equation $y''' - 4y' = 0$.

Or

(b) Compute the solution of $y''' + y'' + y' + y = 1$.

23. (a) Find two power series solutions of $y'' - xy' + y = 0$.

Or

(b) If $\varphi_1, \varphi_2, \ldots \varphi_n$ are n solutions of $L(y)=0$ on I, prove that they are linearly independent if and only if $W(\varphi_1 \ldots \varphi_n)(x) \neq 0$ for all x in I.

24. (a) Derive Bessel function of zero order of the first kind denoted by J_0.

Or

(b) Solve the Euler equation of n^{th} order $x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \ldots a_n y = 0$.

25. (a) For the problems given compute the first four approximations $\varphi_0, \varphi_1, \varphi_2, \varphi_3$

(i) $y' = x^2 + y^2$, $y(0) = 0$

(ii) $y' = 1 + xy$, $y(0) = 1$.

Or

(b) Verify whether the given equations are exact or not if exact solve

(i) $(x+y)dx + (x-y)dy = 0$

(ii) $\cos x \cos^2 y\,dx - \sin x \sin 2y\,dy = 0$.

(8 pages) **Reg. No. :** ....

Code No. : 8156 **Sub. Code : VMAE 13**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2024.

First Semester

Mathematics

Elective — MATHEMATICAL STATISTICS

(For those who joined in July 2024 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. A function which assigns to each element C in $\mathcal{C}$, one and only one real number $X(C)$ is called ————

   (a) Mean (b) Variance

   (c) Random variable (d) Sample

2. The probability of the null set is ————

   (a) 0 (b) 1

   (c) 2 (d) ∞

3. The probability of a set and the probability of its complement is ————

   (a) 0 (b) -1

   (c) 3 (d) 1

4. The value of $P(C_1/C_1)$ is ————.

   (a) -1 (b) 0

   (c) 1 (d) ∞

5. The marginal p.d.f. of x_2 given $X_1 = x_1$ in continuous case is ————.

   (a) $\int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$ (b) $\int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$

   (c) $\int_{-\infty}^{0} f(x_1, x_2)\, dx_1$ (d) $\int_{0}^{\infty} f(x_1, x_2)\, dx_1$

6. If the m.g.f of a binomial distribution X is $(2/3 + 1/3e^t)^6$ then the mean of X is ————.

   (a) 2 (b) 2/3

   (c) 5/3 (d) 4/3

7. The mean of the gamma distribution is ————

   (a) α (b) β

   (c) $\alpha\beta$ (d) 2α

8. If X has the p.d.f. $f(x)=1/4xe^{-x/2}$, $0<X<\infty$, 0 elsewhere then X is ————.

(a) $\chi^2(4)$ (b) $\chi^2(2)$

(c) $\chi^2(1)$ (d) $\chi^3(4)$

9. The Jacobian of $x_1=w_1(y_1,y_2)$, $x_2=w_2(y_1,y_2)$ is

(a) $\frac{\partial x_1}{\partial y_1}\frac{\partial y_2}{\partial y_2}-\frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial x_1}$ (b) $\frac{\partial y_1}{\partial y_1}\frac{\partial x_2}{\partial y_2}-\frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1}$

(c) $\frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2}-\frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1}$ (d) $\frac{\partial y_1}{\partial x_1}\frac{\partial x_2}{\partial y_2}-\frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial x_1}$

10. The distribution W is ———— if V is $\chi^2(r)$.

(a) $n(0,1)$ (b) $n(1,n)$

(c) $n^2(0,1)$ (d) $n(1,1)$

11. The test static for mean is ————

(a) X/n (b) X/N

(c) $\sum_{i=1}^{n}\frac{X_i}{n}$ (d) $\sum_{i=1}^{n}\frac{X_i}{t}$

12. The value of c is ———— if $f(x)=cx(1-x)3$, $0<x<1$, zero elsewhere is a beta p.d.f.

(a) 20 (b) 24

(c) 32 (d) 23

13. Let the distribution function $F_n(x)$ of the random variable Y_n defined upon n a positive integer. If $F(Y)$ is a distribution function and if $\lim_{n\to\infty} F_n(Y)=$ ———— for all y then the random variable Y_n is said to have a limiting distribution.

(a) $F(y)$ (b) $F(tr)$

(c) $F(n)$ (d) F ©

14. If Pr (if $\lim_{n\to\infty} Y_n=c$) = ———— if $Y_n \to c$.

(a) 0 (b) –1

(c) 1 (d) ∞

15. The mean $\overline{X}_n$ of a random sample converges with probability 1 to the mean μ then this form is called strong law of ————

(a) small numbers (b) large numbers

(c) weak numbers (d) convergence

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If C_1 and C_2 are subsets of $\mathcal{C}$ such that $C_1 \subset C_2$ then prove that $P(C_1)\le P(C_2)$.

Or

(b) Let $f(x)=1/3$, $x=-1, 0, 1$ zero elsewhere. Find the distribution function of X.

17. (a) If the m.g.f. of a random variable X is $M(t) = e^{4(e^t-1)}$ then find $P(x = 3)$.

Or

(b) Let the continuous type of random variable X and Y have the joint p.d.f. $f(x, y) = e^{-y}$, $0 < x < y < \infty$ 0 else where. Find m.g.f. and mean.

18. (a) If the random variable X is $n(\mu, \sigma^2)$, $\sigma^2 > 0$ then show that the random variable $W = (X - \mu)/\sigma$ is $n(0, 1)$.

Or

(b) Derive the mean and variance of a gamma distribution.

19. (a) Let the stochastic random variables X_1 and X_2 have the same p.d.f. $f(x) = x/6$, $x = 1, 2, 3, 0$, elsewhere

Find :

(i) $\Pr(X_1 = 2, X_2 = 3)$

(ii) $\Pr(X_1 + X_2 = 3)$.

Or

(b) Let $\overline{X}$ be the mean of the random sample size 5 from a normal distribution with mean 0 and variance 125. Determine c so that $\Pr(\overline{X} < c) = 0.90$.

20. (a) Let $F_n(u)$ denote the distribution function of a random variable U_n whose distribution depends upon the positive integer n. Let U_n converge stochastically to the positive constant c and let $\Pr(Un < 0) = 0$ for every n. Prove that the random variable $\sqrt{U_n}$ converge stochastically to $\sqrt{c}$.

Or

(b) Let X be χ^2 (50) Approximate $\Pr(40 < X < 60)$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

21. (a) State and prove Chebyshev's inequality.

Or

(b) Let X and Y have the p.d.f. $f(x, y) = x + y$ $0 < x < 1$, $0 < y < 1$ 0 elsewhere. Find $E(XY^2)$.

22. (a) Let the random variables X and Y have the joint p.d.f. $f(x, y) = x + y$ 0 elsewhere. Compute the correlation coefficient of X and Y.

Or

(b) If X_1 and X_2 are stochastically independent random variables with m.g.f. $f_1(x_1)$ and $f_2(x_2)$ then Let $M(t_1, t_2)$ denote the m.g.f. of the distribution. Then prove that X_1 and X_2 are stochastically independent if and only if $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$.

23. (a) Find the m.g.f. of a normal distribution.

Or

(b) Let the random variable X have p.d.f. $f(x) = 1$, $0 < x < 1$, 0 elsewhere. Let X_1 and X_2 denote random sample from the distribution. The joint p.d.f. of X_1 and X_2 is $(X_1, X_2) = f(x_1) f(x_2) = 1$. Consider two random variables $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ then find the joint p.d.f. of Y_1 and Y_2.

24. (a) Prove that :

(i) $\overline{X}$ is $n\left(\mu, \frac{\sigma^2}{n}\right)$

(ii) $\frac{ns^2}{\sigma^2}$ is $\chi^2(n-1)$ and

(iii) $\overline{X}$ and S^2 are stochastically independent.

Or

(b) Derive student's 't' distribution.

25. (a) State and prove Central limit theorem.

Or

(b) Let $F_n(y)$ denote the distribution function of a random variable Y_n whose distribution depends upon the positive integer n. Let c denote a constant which does not depend upon n. Prove that the random variable Y_n converges stochastically to the constant c if and only if, for every $\epsilon > 0$ the $\lim_{n \to \infty} \Pr(|Y_n - c| < \epsilon) = 1$.

(8 pages)

Reg. No. : ....

Code No. : 7758 **Sub. Code : WMAE 15/ VMAE 15**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2024.

First Semester

Mathematics — Core

Elective – II – ANALYTIC NUMBER THEORY

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. Given any two integers a and b. For a common divisor d of a and b of the form ———— where $x, y \in Z$.

(a) $d = \frac{ax}{ab}y$ (b) $d = ax - by$

(c) $d = ay + bx$ (d) $ax + by$

2. A reduced fraction is a rational number with

(a) $(a,b) \neq 1$ (b) $(a,b) = 1$

(c) $(a,b) = l, l > 1$ (d) $(a,b) = l, l < 1$

3. If $(a,b) = 1$ then $(a+b, a^2 - ab + b^2)$ is

(a) either 1 or 2 (b) either 1 or 3

(c) 2 (d) 4

4. A real or complex valued function defined on the positive integers are called ———— function.

(a) Arithmetic (b) Logarithmic

(c) Constant (d) Euler

5. If $(m,n) = 1$ then $\phi(mn) =$

(a) $\phi(m) + \phi(n)$ (b) $\phi(m) - \phi(n)$

(c) $\phi(n) - \phi(m)$ (d) $\phi(m)\phi(n)$

6. The arithmetical function $I(n) =$

(a) $\left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 1 \end{cases}$ (b) $[n]$

(c) $1 \forall n \geq 1$ (d) infinity

7. If f is multiplicative then $f(1) =$

(a) 0 (b) 1

(c) −1 (d) infinity

8. $\lambda^{-1}(n) =$

(a) 1 (b) $|\mu(n)|$

(c) −1 (d) ±1

9. A multiplicative function is completely multiplicative if

(a) $f^{-1}(n) = \mu(n)f(n) \forall n \geq 1$

(b) $f^{-1}(n) \neq \mu(n)f(n) \forall n \geq 1$

(c) $f^{-1}(n) > \mu(n)f(n) \forall n \geq 1$

(d) $f^{-1}(n) < \mu(n)f(n) \forall n \geq 1$

10. Average order of $\sigma_1(n)$ is

(a) $\frac{n\pi^2}{6}$ (b) $\frac{n\pi^2}{1^2}$

(c) $\frac{\pi^2}{6}$ (d) $\frac{\pi^n}{6}$

11. The set of lattice points visible from the origin has density

(a) $\frac{6}{\pi^2}$ (b) π^2

(c) $\frac{\pi^2}{6}$ (d) $\frac{3\pi^2}{4}$

12. $\lim_{x\to\infty} \frac{1}{x}\sum_{n\leq x} \mu(n) =$

(a) 1 (b) 0

(c) infinity (d) none of these

13. $\psi(x) =$

(a) $\sum_{n\leq x} \mu(n)$ (b) $\sum_{n\leq x} \lambda(n)$

(c) $\sum_{n\leq x} \Lambda(n)$ (d) $\sum_{n\leq x} \theta(n)$

14. For all $x \geq 1$, $\sum_{p\leq x} \frac{\log p}{p} =$

(a) $\log x + O(1)$ (b) $x\log x + O(x)$

(c) $x\log x + O(x)$ (d) $\log x + O(x)$

15. The series $\sum_{n=1}^{\infty} \frac{1}{p_n}$

(a) converges

(b) diverges

(c) either converges or diverges

(d) none

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let $m = ax + by$ and $n = cx + dy$ where $ad - bc = \pm 1$ prove that $(m, n) = (x, y)$.

Or

(b) State and prove division algorithm.

17. (a) If $n \geq 1$ then $\sum_{d|n} \mu(d) = \left[\frac{1}{n}\right] = \begin{cases} 1 & if \quad n = 1 \\ 0 & if \quad n > 1 \end{cases}$.

Or

(b) Define Mangolt function $\wedge(n)$. Also prove that $\log n = \sum_{d|n} \wedge(d)$ if $n \geq 1$.

18. (a) Define Mobius function. Is mobius function is multiplicative or completely multiplicative? Justify your answer.

Or

(b) Assume that f is multiplicative. Prove that

(i) $f^{-1}(n) = \mu(n) f(n)$ where n is square free

(ii) $f^{-1}(p^2) = \left(f^{-1}(p)\right)^2 - f(p^2)$, for every price p.

19. (a) Deduce the following by using Euler's summation formula $\sum_{n \leq x} \frac{\log n}{x} = \frac{1}{2} \log^2 x + A = O\left(\frac{\log x}{x}\right)$, where A is a constant.

Or

(b) State and prove Euler summation formula.

20. (a) If $h = f * g$ and $H(x) = \sum_{n \leq x} h(n)$,

$F(x) = \sum_{n \leq x} f(n)$ and $G(x) = \sum_{n \leq x} g(n)$ then

$H(x) = \sum_{n \leq x} f(n)\, G\left(\frac{x}{n}\right) = \sum_{n \leq x} g(x) f\left(\frac{x}{n}\right)$.

Or

(b) For $n \geq 1$, then n^{th} prime p_n satisfies the inequality
$\frac{1}{6} n \log n < p_n < 12\left(n \log n + n \log \frac{12}{e}\right)$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)

21. (a) (i) State and prove Euclidean algorithm.

(ii) Prove that $n^4 + 4$ is composite if $n > 1$.

Or

(b) State and prove fundamental theorem of Arithmetic.

22. (a) If $n \geq 1$ then $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$.

Or

(b) (i) $\phi(mn) = \phi(m)\phi(n)\left(\frac{d}{\phi(d)}\right)$ if $d = (m, n)$.

(ii) $\phi(mn) = \phi(m)\phi(n)$ if $(m, n) = 1$.

23. (a) Prove that if f and g are multiplicative so is their dirichlet product. Also, prove that if both g and $f * g$ are multiplicative then f is multiplicative.

Or

(b) State and prove generalized Mobiur inversion formula.

24. (a) For all $x \geq 1$, prove that $\sum_{n \leq x} d(n) = x \log x + (2(-1))x + O(\sqrt{x})$ where C is Euler's constant.

Or

(b) For all $x \geq 1$ and $\alpha > 0$, $\alpha \neq 1$. Prove that

(i) $\sum_{n \leq x} \sigma_1(n) = \frac{1}{2} g(2) x^2 + O(x \log x)$

(ii) $\sum_{n \leq x} \sigma_\alpha(n) = \frac{g(\alpha+1)}{\alpha+1} x^{\alpha+1}$, where $\beta = \max\{1, \alpha\}$.

25. (a) State and prove Legendre identity.

Or

(b) The following are equivalent

(i) $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1$

(ii) $\lim_{x \to \infty} \frac{\gamma(x)}{x} = 1$

(iii) $\lim_{x \to \infty} \frac{\psi(x)}{x} = 1$.